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① a) $A = \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$

i) Verify $A(B+C) = AB + AC$.

$A(B+C)$

$= \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \right)$

$= \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1+3 & -4+2 \\ 2+-1 & 3+1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 & -2 \\ 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} -2 \cdot 4 + 4 \cdot 1 & -2 \cdot -2 + 4 \cdot 4 \\ 1 \cdot 4 + 5 \cdot 1 & 1 \cdot -2 + 5 \cdot 4 \end{bmatrix}$

$= \begin{bmatrix} -8+4 & 4+16 \\ 4+5 & -2+20 \end{bmatrix} = \begin{bmatrix} -4 & 20 \\ 9 & 18 \end{bmatrix}$

$AB + AC$

$= \left(\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \right) + \left(\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \right)$

$= \begin{bmatrix} -2 \cdot 1 + 4 \cdot 2 & -2 \cdot -4 + 4 \cdot 3 \\ 1 \cdot 1 + 5 \cdot 2 & 1 \cdot -4 + 5 \cdot 3 \end{bmatrix} + \begin{bmatrix} -2 \cdot 3 + 4 \cdot -1 & -2 \cdot 2 + 4 \cdot 1 \\ 1 \cdot 3 + 5 \cdot -1 & 1 \cdot 2 + 5 \cdot 1 \end{bmatrix}$

$= \begin{bmatrix} -2+8 & 8+12 \\ 1+10 & -4+15 \end{bmatrix} + \begin{bmatrix} -6+-4 & -4+4 \\ 3+-5 & 2+5 \end{bmatrix}$

$= \begin{bmatrix} 6 & 20 \\ 11 & 11 \end{bmatrix} + \begin{bmatrix} -10 & 0 \\ -2 & 7 \end{bmatrix}$

$= \begin{bmatrix} -4 & 20 \\ 9 & 18 \end{bmatrix}$

Since $\begin{bmatrix} -4 & 20 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} -4 & 20 \\ 9 & 18 \end{bmatrix}$ then $A(B+C) = AB + AC$! :)

ii) Verify $(A+B)(A-B) = A^2 - B^2 + BA - AB$

$(A+B)(A-B)$

$= \left(\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \right) \left(\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \right)$

$= \begin{bmatrix} -2+1 & 4+-4 \\ 1+2 & 5+3 \end{bmatrix} \cdot \begin{bmatrix} -2-1 & 4--4 \\ 1-2 & 5-3 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & 8 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -1 \cdot -3 + 0 \cdot -1 & -1 \cdot 8 + 0 \cdot 2 \\ 3 \cdot -3 + 8 \cdot -1 & 3 \cdot 8 + 8 \cdot 2 \end{bmatrix}$

$= \begin{bmatrix} 3+0 & -8+0 \\ -9+-8 & 24+16 \end{bmatrix}$

$= \begin{bmatrix} 3 & -8 \\ -17 & 40 \end{bmatrix}$

$A^2 - B^2 + BA - AB$

$\left(\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \right) - \left(\begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \right) + \left(\begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \right) - \left(\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \right)$

A^2

$\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -2 \cdot -2 + 4 \cdot 1 & -2 \cdot 4 + 4 \cdot 5 \\ 1 \cdot -2 + 5 \cdot 1 & 1 \cdot 4 + 5 \cdot 5 \end{bmatrix}$

$= \begin{bmatrix} 4+4 & -8+20 \\ -2+5 & 4+25 \end{bmatrix}$

$= \begin{bmatrix} 8 & 12 \\ 3 & 29 \end{bmatrix}$

B^2

$\begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + -4 \cdot 2 & 1 \cdot -4 + -4 \cdot 3 \\ 2 \cdot 1 + 3 \cdot 2 & 2 \cdot -4 + 3 \cdot 3 \end{bmatrix}$

$= \begin{bmatrix} 1+-8 & -4+-12 \\ 2+6 & -8+9 \end{bmatrix}$

$= \begin{bmatrix} -7 & -16 \\ 8 & 1 \end{bmatrix}$

BA

$\begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot -2 + -4 \cdot 1 & 1 \cdot 4 + -4 \cdot 5 \\ 2 \cdot -2 + 3 \cdot 1 & 2 \cdot 4 + 3 \cdot 5 \end{bmatrix}$

$= \begin{bmatrix} -2+-4 & 4+-20 \\ -4+3 & 8+15 \end{bmatrix}$

$= \begin{bmatrix} -6 & -16 \\ -1 & 23 \end{bmatrix}$

$AB = \begin{bmatrix} 6 & 20 \\ 11 & 11 \end{bmatrix}$

↳ From i)

Summary

$A^2 - B^2 + BA - AB$

$= \begin{bmatrix} 8 & 12 \\ 3 & 29 \end{bmatrix} - \begin{bmatrix} -7 & -16 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} -6 & -16 \\ -1 & 23 \end{bmatrix} - \begin{bmatrix} 6 & 20 \\ 11 & 11 \end{bmatrix}$

$= \begin{bmatrix} 15 & 28 \\ -5 & 28 \end{bmatrix} + \begin{bmatrix} -12 & -36 \\ -12 & 12 \end{bmatrix}$

$= \begin{bmatrix} 3 & -8 \\ -17 & 40 \end{bmatrix}$

Since $\begin{bmatrix} 3 & -8 \\ -17 & 40 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ -17 & 40 \end{bmatrix}$ then $(A+B)(A-B) = A^2 - B^2 + BA - AB$!

iii) $(ABC)^T = C^T B^T A^T$

$$\left(\begin{bmatrix}-2&4\\1&5\end{bmatrix}\cdot\begin{bmatrix}1&-4\\2&3\end{bmatrix}\cdot\begin{bmatrix}3&2\\-1&1\end{bmatrix}\right)^T$$

$$=\left(\begin{bmatrix}-2\cdot1+4\cdot2 & -2\cdot-4+4\cdot3\\1\cdot1+5\cdot2 & 1\cdot-4+5\cdot3\end{bmatrix}\cdot\begin{bmatrix}3&2\\-1&1\end{bmatrix}\right)^T$$

$$=\left(\begin{bmatrix}-2+8 & 8+12\\1+10 & -4+15\end{bmatrix}\cdot\begin{bmatrix}3&2\\-1&1\end{bmatrix}\right)^T$$

$$=\left(\begin{bmatrix}6 & 20\\11 & 11\end{bmatrix}\cdot\begin{bmatrix}3&2\\-1&1\end{bmatrix}\right)^T$$

$$=\left(\begin{bmatrix}6\cdot3+20\cdot-1 & 6\cdot2+20\cdot1\\11\cdot3+11\cdot-1 & 11\cdot2+11\cdot1\end{bmatrix}\right)^T$$

$$=\begin{bmatrix}18+-20 & 12+20\\33+-11 & 22+11\end{bmatrix}^T$$

$$=\begin{bmatrix}-2 & 32\\22 & 33\end{bmatrix}^T=\begin{bmatrix}-2 & 22\\32 & 33\end{bmatrix}$$

$$\begin{bmatrix}3&2\\-1&1\end{bmatrix}^T\cdot\begin{bmatrix}1&-4\\2&3\end{bmatrix}^T\cdot\begin{bmatrix}-2&4\\1&5\end{bmatrix}^T$$

$$=\begin{bmatrix}3&-1\\2&1\end{bmatrix}\cdot\begin{bmatrix}1&2\\-4&3\end{bmatrix}\cdot\begin{bmatrix}-2&1\\4&5\end{bmatrix}$$

$$=\begin{bmatrix}3\cdot1+-1\cdot-4 & 3\cdot2+-1\cdot3\\2\cdot1+1\cdot-4 & 2\cdot2+1\cdot3\end{bmatrix}\cdot\begin{bmatrix}-2&1\\4&5\end{bmatrix}$$

$$=\begin{bmatrix}3+4 & 6+-3\\2+-4 & 4+3\end{bmatrix}\cdot\begin{bmatrix}-2&1\\4&5\end{bmatrix}$$

$$=\begin{bmatrix}7&3\\-2&7\end{bmatrix}\cdot\begin{bmatrix}-2&1\\4&5\end{bmatrix}$$

$$=\begin{bmatrix}7\cdot-2+3\cdot4 & 7\cdot1+3\cdot5\\-2\cdot-2+7\cdot4 & -2\cdot1+7\cdot5\end{bmatrix}$$

$$=\begin{bmatrix}-14+12 & 7+15\\4+28 & -2+35\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 22\\32 & 33\end{bmatrix}$$

Since $\begin{bmatrix}-2&22\\32&33\end{bmatrix}=\begin{bmatrix}-2&22\\32&33\end{bmatrix}$, therefore, $(ABC)^T = C^T B^T A^T$

b) i) $2A-3B-C$

$$=2\cdot\begin{bmatrix}-2&4\\1&5\end{bmatrix}-3\cdot\begin{bmatrix}1&-4\\2&3\end{bmatrix}-\begin{bmatrix}3&2\\-1&1\end{bmatrix}$$

$$=\begin{bmatrix}2\cdot-2 & 2\cdot4\\2\cdot1 & 2\cdot5\end{bmatrix}-\begin{bmatrix}3\cdot1 & 3\cdot-4\\3\cdot2 & 3\cdot3\end{bmatrix}-\begin{bmatrix}3&2\\-1&1\end{bmatrix}$$

$$=\begin{bmatrix}-4&8\\2&10\end{bmatrix}-\begin{bmatrix}3&-12\\6&9\end{bmatrix}-\begin{bmatrix}3&2\\-1&1\end{bmatrix}$$

$$=\begin{bmatrix}-4-3 & 8--12\\2-6 & 10-9\end{bmatrix}-\begin{bmatrix}3&2\\-1&1\end{bmatrix}$$

$$=\begin{bmatrix}-7&20\\-4&1\end{bmatrix}-\begin{bmatrix}3&2\\-1&1\end{bmatrix}$$

$$=\begin{bmatrix}-7-3 & 20-2\\-4--1 & 1-1\end{bmatrix}$$

$$=\begin{bmatrix}-10&18\\-3&0\end{bmatrix}$$

$$2A-3B-C=\begin{bmatrix}-10&18\\-3&0\end{bmatrix}$$

$(A-2B)(C+3B)$

$$=\left(\begin{bmatrix}-2&4\\1&5\end{bmatrix}-2\begin{bmatrix}1&-4\\2&3\end{bmatrix}\right)\left(\begin{bmatrix}3&2\\-1&1\end{bmatrix}+3\begin{bmatrix}1&-4\\2&3\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}-2&4\\1&5\end{bmatrix}-\begin{bmatrix}2&-8\\4&6\end{bmatrix}\right)\left(\begin{bmatrix}3&2\\-1&1\end{bmatrix}+\begin{bmatrix}3&-12\\6&9\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}-2-2 & 4--8\\1-4 & 5-6\end{bmatrix}\right)\left(\begin{bmatrix}3+3 & 2+-12\\-1+6 & 1+9\end{bmatrix}\right)$$

$$=\begin{bmatrix}-4&12\\-3&-1\end{bmatrix}\cdot\begin{bmatrix}6&-10\\5&10\end{bmatrix}$$

$$=\begin{bmatrix}-4\cdot6+12\cdot5 & -4\cdot-10+12\cdot10\\-3\cdot6+-1\cdot5 & -3\cdot-10+-1\cdot10\end{bmatrix}$$

$$=\begin{bmatrix}-24+60 & 40+120\\-18+-5 & 30+-10\end{bmatrix}$$

$$=\begin{bmatrix}36&160\\-23&20\end{bmatrix}$$

$$(A-2B)(C+3B)=\begin{bmatrix}36&160\\-23&20\end{bmatrix}$$

2) $A = \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix}$ $B = (4 \ -2)$ $C = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

column

rows $\leftarrow m \times \underbrace{n \quad n}_{\text{Equal.}} \times p \rightarrow m \times p$

Dimension.

a) AC

$= \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 3 \cdot -1 + 2 \cdot 5 \\ -2 \cdot -1 + 4 \cdot 5 \end{bmatrix}$

$= \begin{bmatrix} -3 + 10 \\ 2 + 20 \end{bmatrix}$

$= \begin{bmatrix} 7 \\ 22 \end{bmatrix}$

b) CA

↳ Invalid...

= Different dimensions.

$C = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ $A = \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix}$

$2 \times \underbrace{1 \qquad 2}_{} \times 2$

Cannot multiply $1 \neq 2$.

c) $A(B^T + C)$

$= \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix} \cdot \left([4 \ -2]^T + \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right)$

$= \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} \right)$

$= \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 3 \cdot 3 + 2 \cdot 3 \\ -2 \cdot 3 + 4 \cdot 3 \end{bmatrix}$

$= \begin{bmatrix} 9 + 6 \\ -6 + 12 \end{bmatrix}$

$= \begin{bmatrix} 15 \\ 6 \end{bmatrix}$

c) BC.

$[4 \ -2] \cdot \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$1 \times \underbrace{2 \quad 2}_{\checkmark} \times 1$

$= [4 \cdot -1 + -2 \cdot 5]$

$= [-4 + -10]$

$= [-14]$

d) $C^T B^T$

$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}^T \cdot [4 \ -2]^T$

$= [-1 \ 5] \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$= [-1 \cdot 4 + 5 \cdot -2]$

$= [-4 + -10]$

$= [-14]$

f) BB^T

$= [4 \ -2] \cdot [4 \ -2]^T$

$= [4 \ -2] \cdot \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$= [4 \cdot 4 + -2 \cdot -2]$

$= [16 + 4]$

$= [20]$

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3) $A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

a) $C^T A$

$= \begin{bmatrix} -1 \\ 2 \end{bmatrix}^T \cdot \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix}$

$= [-1 \ 2] \cdot \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix}$

$1 \times \underbrace{2 \quad 2}_{\text{equal.}} \times 3$

$= [-1 \cdot 3 + 2 \cdot -2 \quad -1 \cdot 1 + 2 \cdot 4 \quad -1 \cdot 2 + 2 \cdot -2]$

$= [-3 + -4 \quad -1 + 8 \quad -2 + -4]$

$= [-7 \quad 7 \quad -6]$

b) $A^T C$

$= \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix}^T \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 \\ 1 & 4 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$m \times n \quad n \times p$

$3 \times \underbrace{2 \quad 2}_{\text{equal.}} \times 1$

$m \times p$.

3×1

$(3 \cdot -1) + (-2 \cdot 2)$

$= -3 + -4$

$= -7$

$(1 \cdot -1) + (4 \cdot 2)$

$= -1 + 8$

$= 7$

$(2 \cdot -1) + (-2 \cdot 2)$

$= -2 + -4$

$= -6$

$\longrightarrow A^T C = \begin{bmatrix} -7 \\ 7 \\ -6 \end{bmatrix} \longleftarrow$

c) $AA^T BC$

$= \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix}^T \cdot \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ 1 & 4 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= \left(\begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ 1 & 4 \\ 2 & -2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right)$

$2 \times \underbrace{3 \quad 3}_{} \times 2 \qquad 2 \times \underbrace{2 \quad 2}_{} \times 1$

$= \begin{bmatrix} 3(3) + 1(1) + 2(2) & 3(-2) + 1(4) + 2(-2) \\ -2(3) + 4(1) + -2(2) & -2(-2) + 4(4) + -2(-2) \end{bmatrix}$

$\cdot \begin{bmatrix} 2(-1) + -1(2) \\ 3(-1) + 4(2) \end{bmatrix}$

$= \begin{bmatrix} 14 & -6 \\ -6 & 24 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 14(-4) + -6(5) \\ -6(-4) + 24(5) \end{bmatrix}$

$= \begin{bmatrix} -86 \\ 144 \end{bmatrix}$

$AA^T BC = \begin{bmatrix} -86 \\ 144 \end{bmatrix}$.

4)

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 4 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

a) $(A-B)(A+B)$

$$\left(\begin{bmatrix} 3 & 2 & -2 \\ 4 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}\right)\left(\begin{bmatrix} 3 & 2 & -2 \\ 4 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3-1 & 2-(-2) & -2-0 \\ 4-2 & (-1)-1 & 2-(-1) \\ -2-1 & 0-(-3) & -1-2 \end{bmatrix} \cdot \begin{bmatrix} 3+1 & 2+(-2) & -2+0 \\ 4+2 & -1+1 & 2+(-1) \\ -2+1 & 0+(3) & -1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -2 \\ 2 & -2 & 3 \\ -3 & 3 & -3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & -2 \\ 6 & 0 & 1 \\ -1 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(4)+4(6)+-2(-1) & 2(0)+4(0)+-2(-3) & 2(-2)+4(1)+-2(1) \\ 2(4)+-2(6)+3(-1) & 2(0)+-2(0)+3(-3) & 2(-2)+-2(1)+3(1) \\ -3(4)+3(6)+-3(-1) & -3(0)+3(0)+-3(-3) & -3(-2)+3(1)+-3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 8+24+2 & 0+0+6 & -4+4+-2 \\ 8+-12+-3 & 0+0+-9 & -4+-2+3 \\ -12+18+3 & 0+0+9 & 6+3+-3 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 6 & -2 \\ -7 & -9 & -3 \\ 9 & 9 & 6 \end{bmatrix}$$

So, $(A-B)(A+B) = \begin{bmatrix} 36 & 6 & -2 \\ -7 & -9 & -3 \\ 9 & 9 & 6 \end{bmatrix}$

b) $A^2 - B^2$

$$\left(\begin{bmatrix} 3 & 2 & -2 \\ 4 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & -2 \\ 4 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix}\right) - \left(\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3(3)+2(4)+-2(-2) & 3(2)+2(-1)+-2(0) & 3(-2)+2(2)+-2(-1) \\ 4(3)+-1(4)+2(-2) & 4(2)+-1(-1)+2(0) & 4(-2)+-1(2)+2(-1) \\ -2(3)+0(4)+-1(-2) & -2(2)+0(-1)+-1(0) & -2(-2)+0(2)+-1(-1) \end{bmatrix} -$$

$$\begin{bmatrix} 1(1)+-2(2)+0(-1) & 1(-2)+-2(1)+0(3) & 1(0)+-2(-1)+0(2) \\ 2(1)+1(2)+-1(1) & 2(-2)+1(1)+-1(3) & 2(0)+1(-1)+-1(2) \\ 1(1)+-3(2)+2(1) & 1(-2)+-3(1)+2(3) & 1(0)+-3(-1)+2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 9+8+4 & 6+-2+0 & -6+4+2 \\ 12+-4+-4 & 8+1+0 & -8+-2+-2 \\ -6+0+2 & -4+0+0 & 4+0+-1 \end{bmatrix} - \begin{bmatrix} 1+-4+0 & -2+-2+0 & 0+2+0 \\ 2+2+-1 & -4+1+3 & 0+-1+-2 \\ 1+-6+2 & -2+-3+-6 & 0+3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 4 & 0 \\ 4 & 9 & -12 \\ -4 & -4 & 5 \end{bmatrix} - \begin{bmatrix} -3 & -4 & 2 \\ 3 & 0 & -3 \\ -3 & -11 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 8 & -2 \\ 1 & 9 & -9 \\ -1 & 7 & -2 \end{bmatrix}$$

Therefore, $A^2 - B^2 = \begin{bmatrix} 24 & 8 & -2 \\ 1 & 9 & -9 \\ -1 & 7 & -2 \end{bmatrix}$

c) $AB - BA$

$$\left(\begin{bmatrix} 3 & 2 & -2 \\ 4 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}\right) - \left(\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & -2 \\ 4 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3(1)+2(2)+-2(1) & 3(-2)+2(1)+-2(-3) & 3(0)+2(-1)+-2(2) \\ 4(1)+-1(2)+2(1) & 4(-2)+-1(1)+2(-3) & 4(0)+-1(-1)+2(2) \\ -2(1)+0(2)+-1(1) & -2(-2)+0(1)+-1(-3) & -2(0)+0(1)+-1(2) \end{bmatrix} -$$

$$\begin{bmatrix} 1(3)+-2(4)+0(-2) & 1(2)+-2(-1)+0(0) & 1(-2)+-2(2)+0(-1) \\ 2(3)+1(4)+-1(-2) & 2(2)+1(-1)+-1(0) & 2(-2)+1(2)+-1(-1) \\ 1(3)+-3(4)+2(-2) & 1(2)+-3(-1)+2(0) & 1(-2)+-3(2)+2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3+4+-2 & -6+2+6 & 0+-2+-4 \\ 4+-2+2 & -8+-1+-6 & 0+1+4 \\ -2+0+-1 & 4+0+3 & 0+0+2 \end{bmatrix} - \begin{bmatrix} 3+-8+0 & 2+2+0 & -2+-4+0 \\ 6+4+2 & 4+-1+0 & -4+2+1 \\ 3+-12+-4 & 2+3+0 & -2+-6+-2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & -6 \\ 4 & -15 & 5 \\ -3 & 7 & 2 \end{bmatrix} - \begin{bmatrix} -5 & 4 & -6 \\ 12 & 3 & -1 \\ -13 & 5 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -2 & 0 \\ -8 & -18 & 6 \\ 10 & 2 & 8 \end{bmatrix}$$ Therefore, $AB - BA = \begin{bmatrix} 10 & -2 & 0 \\ -8 & -18 & 6 \\ 10 & 2 & 8 \end{bmatrix}$

d) $A^T B + B^T A$

$$\left(\begin{bmatrix} 3 & 4 & -2 \\ 2 & -1 & 0 \\ -2 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}\right) + \left(\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -3 \\ 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & -2 \\ 4 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3(1)+4(2)+-2(1) & 3(-2)+4(1)+-2(-3) & 3(0)+4(-1)+-2(2) \\ 2(1)+-1(2)+0(1) & 2(-2)+-1(1)+0(-3) & 2(0)+-1(-1)+0(2) \\ -2(1)+2(2)+-1(1) & -2(-2)+2(1)+-1(-3) & -2(0)+2(-1)+-1(2) \end{bmatrix} +$$

$$\begin{bmatrix} 1(3)+2(4)+1(-2) & 1(2)+2(-1)+1(0) & 1(-2)+2(2)+1(-1) \\ -2(3)+1(4)+-3(-2) & -2(2)+1(-1)-3(0) & -2(-2)+1(2)-3(-1) \\ 0(3)+-1(4)+2(-2) & 0(2)+-1(-1)+2(0) & 0(-2)+-1(2)+2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3+8+-2 & -6+4+6 & 0+-4+-4 \\ 2+-2+0 & -4+-1+0 & 0+1+0 \\ -2+4+-1 & 4+2+3 & 0+-2+-2 \end{bmatrix} + \begin{bmatrix} 3+8+-2 & 2+-2+0 & -2+4-1 \\ -6+4+6 & -4+-1+0 & 4+2+3 \\ 0+-4+-4 & 0+1+0 & 0+-2+-2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 4 & -8 \\ 0 & -5 & 1 \\ 1 & 9 & -4 \end{bmatrix} + \begin{bmatrix} 9 & 0 & -1 \\ 4 & -5 & 9 \\ -8 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 & -7 \\ 4 & -10 & 10 \\ -7 & 10 & -8 \end{bmatrix}$$ $A^T B + B^T A = \begin{bmatrix} 18 & 4 & -7 \\ 4 & -10 & 10 \\ -7 & 10 & -8 \end{bmatrix}$

(5) For any $m \times n$ matrices

a) $A + B = B + A$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix}$$

Let A_{ij}, B_{ij} be an element in A & B where i = row, j = columns.

LHS	RHS.
$A + B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1n}+B_{1n} \\ A_{21}+B_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1}+B_{n1} & \cdots & & A_{nn}+B_{nn} \end{bmatrix}$	$B + A = \begin{bmatrix} B_{11}+A_{11} & B_{12}+A_{12} & \cdots & B_{1n}+A_{1n} \\ B_{21}+A_{21} & \ddots & & \\ \vdots & & \ddots & \\ B_{n1}+A_{n1} & \cdots & & B_{1n}+A_{1n} \end{bmatrix}$
For each element is $A + B$, it is $A_{ij} + B_{ij}$	For each element $B + A$, it is $B_{ij} + A_{ij}$

Let $x, y \in \mathbb{R}$

$x + y = y + x$

Since $A_{ij}, B_{ij} \in \mathbb{R}$

Then $A_{ij} + B_{ij} = B_{ij} + A_{ij}$

So $A + B = B + A$

b) $$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1} & \cdots & & B_{nn} \end{bmatrix} \quad C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ C_{n1} & \cdots & & C_{nn} \end{bmatrix}$$

Let A_{ij}, B_{ij}, C_{ij} be an element in A, B, C where i = row j = columns.

$$A + (B + C) = \begin{bmatrix} A_{11}+(B_{11}+C_{11}) & A_{12}+(B_{12}+C_{12}) & \cdots & A_{1n}+(B_{1n}+C_{1n}) \\ A_{21}+(B_{21}+C_{21}) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1}+(B_{n1}+C_{n1}) & \cdots & & A_{nn}+(B_{nn}+C_{nn}) \end{bmatrix}$$

For each element is $A + (B + C)$, it is $A_{ij} + (B_{ij} + C_{ij})$

$$(A + B) + C = \begin{bmatrix} (A_{11}+B_{11})+C_{11} & (A_{12}+B_{12})+C_{12} & \cdots & (A_{1n}+B_{1n})+C_{1n} \\ (A_{21}+B_{21})+C_{21} & \ddots & & \vdots \\ (A_{n1}+B_{n1})+C_{n1} & \cdots & \ddots & \vdots \\ & & & (A_{nn}+B_{nn})+C_{nn} \end{bmatrix}$$

For each element is $(A + B) + C$, it is $(A_{ij} + B_{ij}) + C_{ij}$

If $x, y, z \in \mathbb{R}$, then $x + (y + z) = (x + y) + z$.

Since $A_{ij}, B_{ij}, C_{ij} \in \mathbb{R}$

So $A + (B + C) = (A + B) + C$

5) c) $\lambda(A+B) = \lambda A + \lambda B$
(continued)

Let $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \ddots & & \\ \vdots & & & \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix}$ Let $B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ B_{n1} & \cdots & & B_{nn} \end{bmatrix}$

Let A_{ij}, B_{ij} represent A, B where i = row, j = columns.

LHS

$$\lambda(A+B) = \begin{bmatrix} \lambda(A_{11}+B_{11}) & \lambda(A_{12}+B_{12}) & \cdots & \lambda(A_{1n}+B_{1n}) \\ \lambda(A_{21}+B_{21}) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda(A_{n1}+B_{n1}) & \cdots & & \lambda(A_{nn}+B_{nn}) \end{bmatrix}$$

For each element in $\lambda(A+B)$, it is $\lambda(A_{ij}+B_{ij})$

RHS

$$\lambda A + \lambda B = \begin{bmatrix} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1n} \\ \lambda A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ \lambda A_{n1} & \cdots & & \lambda A_{nn} \end{bmatrix} + \begin{bmatrix} \lambda B_{11} & \lambda B_{12} & \cdots & \lambda B_{1n} \\ \lambda B_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda B_{n1} & \cdots & & \lambda B_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda A_{11}+\lambda B_{11} & \lambda A_{12}+\lambda B_{12} & \cdots & \lambda A_{1n}+\lambda B_{1n} \\ \lambda A_{21}+\lambda B_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \lambda A_{n1}+\lambda B_{n1} & \cdots & & \lambda A_{nn}+\lambda B_{nn} \end{bmatrix}$$

For each element in $\lambda A + \lambda B$, it is $\lambda A_{ij} + \lambda B_{ij}$

Let $x, y, z \in \mathbb{R}$, $x(y+z) = xy + xz$

Since $\lambda, A_{ij}, B_{ij} \in \mathbb{R}$

$\therefore \lambda(A_{ij}+B_{ij}) = \lambda \cdot A_{ij} + \lambda \cdot B_{ij}$

So $\lambda(A+B) = \lambda A + \lambda B$

6) Find x & y such that $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 2x - y \\ -3x + 4y \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 2x - y + 8 \\ -3x + 4y + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= 2x - y + 8 = 0$

$-3x + 4y + 1 = 0$

$= (4 \cdot 2x) + (4 \cdot -y) + (4 \cdot 8) = 0$

$-3x + 4y + 1 = 0.$

$= 8x \cancel{- 4y} + 32 = 0$

$+ \quad -3x \cancel{+ 4y} + 1 = 0$

$5x \cancel{+ 33} = 0 - 33$

$\cancel{-33}$

$= \frac{5x}{5} = \frac{-33}{5}$

$x = -\frac{33}{5} = -6.6$

Then...

$-3(-\frac{33}{5}) + 4y + 1 = 0$

$\cancel{20.8} + 4y = \frac{0 - 20.8}{4}$

$\cancel{-20.8} \quad 4$

$y = 5.2$

Answer

$x = -6.6$

$y = 5.2$

7) If A & B are square matrices such that $AB = 0$, prove that we can have $A \neq 0$ & $B \neq 0$

Is the result true for non-squared matrices?

(i) Let A = square matrix that $AB = 0$.

Case 1:

A is invertible and $AB = 0$

Then $B = A^{-1}AB$

$= A^{-1}0$

$= 0$

$\therefore B = 0$

Case 2: A is singular. & $\mathbb{F}$ field.

Since A is singular; $\underbrace{\ker(A)}_{\text{kernel of } A} = \{x \in \mathbb{F}^n : Ax = 0\}$

Let $w^T \in \mathbb{F}^{1\times n}$ be any nonzero row vector..

$B = vw^T \in M_n(\mathbb{F})$

$ABx = A(vw^Tx)$

$= (Av)(w^Tx)$

$= 0 \cdot (w^Tx)$

$= 0$

(ii) Non-square matrices.

Let $A \in M_{m\times n}(\mathbb{F})$,

$B \in M_{n\times p}(\mathbb{F})$

- If $\ker(A) \neq 0$, then nonzero B with $AB = 0$ is true.
- If $\ker(A) = 0$ (has full column rank).

Then for $AB = 0$, $B = 0$.

Extra Notes on null space.

Kernel: Set of all input vectors that are mapped to the zero vector by A.

Trivial Kernel

Only zero vector solves $Ax = 0$.

Vectors are pointed to different directions.

$\ker(A) = 0$,

- columns of A are linearly independent.
- Square mat. $A \to$ invertible.

Non-trivial kernel

Non-zero solutions to $Ax = 0$.

- colums of A are linearly dependent.
- A is singular

$\hookrightarrow$ not invertible

$\ker(A) \supsetneq \{0\}$

kernel contains infinitely many non-zero vectors.

* Notes

- Full column rank:

Is when $\mathrm{rank}(A) = n$.

dimension of its columnspace (Span of columns)

- Non-full column rank.

Is when $\mathrm{rank}(A) < n$.

8) If $AB = AC$, is $B = C$?

Let A, B, C be any matrices, such that $AB = AC$ & $B = C$.

$AB = AC = A(B - C) = 0$

$= B - C = 0$?

- If A is invertible,

$A(B - C) = 0$

$B - C = 0$

$B = C$

- If A is not invertible, A is a non-trivial kernel.

Therefore, $B \neq C$.

9) If A, B and C are any square matrices of the same order, prove that... (generalize the results)

a. $A(BC) = (AB)C$

This is the "Associativity" theorem.

Let $X \in \mathbb{R}^{n \times n}$; x_j where j represents columns.

For any matrix M, each column is Mx_j:

$$(A(BC))x_j = A((BC)x_j) = A(B(Cx_j)) \quad \text{(LHS)}$$

and

$$((AB)C)x_j = (AB)(Cx_j) = A(B(Cx_j)) \quad \text{(RHS)}$$

Therefore, $A(BC) = (AB)C$

b. $A(B+C) = AB + AC$.

For each j column...

$$(A(B+C))x_j = A((B+C)x_j)$$
$$= A(Bx_j + Cx_j) \leftarrow \text{distributivity.}$$
$$= A(Bx_j) + A(Cx_j)$$

$\therefore A(B+C) = AB + AC$.

c. $(ABC)^T = C^T B^T A^T$

This is the "Transpose of a product".

Let X, Y be any matrix..

For any vectors, u, v in the vector spaces...

$$\langle Xu, v \rangle = \langle u, X^T v \rangle,$$

$$\langle (XY)u, v \rangle = \langle Yu, X^T v \rangle$$
$$= \langle u, Y^T (X^T v) \rangle$$
$$= \langle u, (Y^T X^T) v \rangle$$

Since, $(XY)^T = Y^T X^T$

Therefore, $(ABC)^T = C^T B^T A^T$

Generalizations.

1. Associative.

$$A_1(A_2(\dots A_n)) = \dots = ((A_1 A_2) \dots) A_n.$$

2. Distributive.

$$A\left(\Sigma_i B_i\right) = \Sigma_i AB_i \quad \left(\Sigma_i B_i\right) A = \Sigma_i B_i A$$

3. Transpose order reverse.

$$(A_1 A_2 \dots A_n)^T = A_n^T \dots A_2^T A_1^T.$$

10) a) $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Show that the transformation can be written $Y = AX$

Given form: $Y = AX$

Since we are transforming X with A, that is essentially multiplying matrix X with matrix A, that is...

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{bmatrix}.$$

Therefore, the transformation can be written as

$Y = AX$

b) $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Show that $x_1 = b_{11} u_1 + b_{11} u_2$
$x_2 = b_{21} u_1 + b_{22} u_2$

x_1 and x_2 can be shown via matrix multiplication.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} \cdot u_1 + b_{12} \cdot u_2 \\ b_{21} \cdot u_1 + b_{22} \cdot u_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} u_1 + b_{12} \cdot u_2 \\ b_{21} u_1 + b_{22} \cdot u_2 \end{bmatrix}$$

Can be re-written as...

$x_1 = b_{11} u_1 + b_{12} u_2$

$x_2 = b_{21} u_1 + b_{22} \cdot u_2.$

→ Obtain y_1, y_2 in terms of u_1, u_2 & explain how you can use the approach to motivate a definition of AB.

y_1, y_2 can be shown through substitution as follows

Since $Y = AX$ and $X = BU$

Then...

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} u_1 + b_{12} u_2 \\ b_{21} u_1 + b_{22} u_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11} u_1 + b_{12} u_2) + a_{12}(b_{21} u_1 + b_{22} u_2) \\ a_{21}(b_{11} u_1 + b_{12} u_2) + a_{22}(b_{21} u_1 + b_{22} u_2) \end{bmatrix}$$

Can also be written as...

$Y = A(BU) \rightarrow (AB)U$

11. Generalize problem 10 to 3 or more dimensions.

i) 3 dimensions.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Transformation ① - Y

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix} \quad \text{or } Y = AX$$

Transformation ② - X

Let $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 \\ b_{21}x_1 + b_{22}x_2 + b_{23}x_3 \\ b_{31}x_1 + b_{32}x_2 + b_{33}x_3 \end{pmatrix}$$

Substitution:

$$y_j = \sum_{i=1}^{3} a_{ji}x_i = \sum_{i=1}^{3} a_{ji}\left(\sum_{k=1}^{3} b_{ik}u_k\right) = \sum_{k=1}^{3}\left(\sum_{i=1}^{3} a_{ji}b_{ik}\right)u_k$$

(Y) (Sum of all A & X) (Sum of all A(BU)) (Sum of all (AB)U)

So: $Y = ABU$

ii) n-dimensions.

For any $n \geq 2$:

- Let X, Y, U be column vectors in $\mathbb{R}^n$.
- Let A, B be $n \times n$ matrices.

Transformations:

$Y = AX$, $X = BU$

$= Y = A(BU) = (AB)U$

can also be written as:

$$y_j = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ji}b_{ik}\right)u_k.$$

Therefore,

$$(AB)_{jk} = \sum_{i=1}^{n} a_{jk}b_{ik}.$$

Generalization.

3-dimensions.

- Transformation.

$$y_j = \sum_{i=1}^{3} a_{ji}x_i$$

- Matrix Entry.

$$(AB)_{jk} = \sum_{i=1}^{3} a_{ji}b_{ik}$$

n-dimensions.

- Transformation

$$y_j = \sum_{i=1}^{n} a_{ji}x_i$$

- Matrix Entry.

$$(AB)_{jk} = \sum_{i=1}^{n} a_{ji}b_{ik}.$$

12.

a) Prove that the relationship, between the coordinates or transformation from (x,y) to (x',y') is given by..

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Let i, j be the unit vectors of the original x, y axes

i', j' be the unit vectors of the rotated x', y' axes.

If x' axis = rotating x-axis counterclockwise by angle θ

then...

$$i' = \cos\theta i + \sin\theta j, \quad j' = -\sin\theta i + \cos\theta j$$

Point P has same geometric position, therefore..

$$\overrightarrow{OP} = xi + yj = x'i' + y'j'$$

$$= xi + yi = x'(\cos\theta i + \sin\theta j) + y'(-\sin\theta i + \cos\theta j)$$

So:

- $x = x'\cos\theta - y'\sin\theta$
- $y = x'\sin\theta + y'\cos\theta$

$$x\cos\theta + y\sin\theta = x'(\cos^2\theta + \sin^2\theta) = x'$$

$$-x\sin\theta + y\cos\theta = y'(\sin^2\theta + \cos^2\theta) = y'$$

$\therefore x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

So, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

* $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the transpose / inverse of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

b) Matrix A is skew-symmetric if $A^T = -A$

Since

$R(\theta)^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $R(\theta)^T \neq -R(\theta)$,

Therefore

the rotation matrix is NOT skew-symmetric.

13) Let vector $\vec{v}$ has (x_1, y_1)

Rotate the vector counterclockwise through angle θ

$\downarrow$

$(x_1, y_1) \rightarrow (x_2, y_2)$.

$\hookrightarrow$ After rotation...

$$x_2 = x_1 \cos\theta - y_1 \sin\theta$$
$$y_2 = x_1 \sin\theta + y_1 \cos\theta$$

Can be written in Matrix Form as:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Problem 12: Axes are rotated by θ while the vector stays fixed.

vs 13: Vector is rotated by θ while axes stay fixed.

14) Let $A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

a) $A(\theta_1 + \theta_2) = A(\theta_1) A(\theta_2)$

$$A(\theta_1)A(\theta_2) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}$$

$$= A(\theta_1 + \theta_2)$$

b) $A(\theta_1 + \theta_2 + \dots + \theta_n) = A(\theta_1)A(\theta_2)\dots A(\theta_n)$

$$A(\theta_1 + \dots + \theta_n) = A((\theta_1 + \dots + \theta_{n-1}) + \theta_n)$$
$$= A(\theta_1 + \dots + \theta_{n-1}) A(\theta_n)$$
$$= A(\theta_1) \dots A(\theta_n)$$

c) $[A(\theta)]^n = A(n\theta) \rightarrow$ Take all θ_k equal to the same θ in part (b):

$$A(\theta + \theta + \dots + \theta) = A(n\theta)$$
$$= A(\theta)A(\theta)\dots A(\theta)$$
$$= [A(\theta)]^n.$$

Discuss:

- These formulas state that rotating by θ_2 and then by θ_1 is essentially rotating once by the sum $\theta_1 + \theta_2$.
- For both Problem 12 & 13, there is a consistent algebraic rule: Composing two rotations simply adds their angles.