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Q1) Eigenvalues of a 2×2 matrix

$$M_1 = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}$$

a) Characteristic equation of $M1$.

Characteristic polynomial.

$$\det(\lambda I - M_1) = (\lambda - A)(\lambda - D) - BC$$
$$= \lambda^2 - (A + D)\lambda + (AD - BC)$$
$$= \lambda^2 - (4+6)\lambda + (4\cdot 6 - 0\cdot 9)$$
$$= \lambda^2 - 10\lambda + 24$$

So, $\lambda^2 - 10\lambda + 24$
↳ Characteristic equation.

b) Eigenvalues.

Solve $\lambda^2 - 10\lambda + 24 = 0$

$\Delta = (-10)^2 - 4\cdot 1\cdot 24$ $\quad \sqrt{\Delta} = 2$
$= 100 - 96$
$= 4,$

Eigenvalues.

$$\lambda = \frac{10 \pm 2}{2} \Rightarrow \lambda_1 = \frac{12}{2} = 6, \quad \lambda_2 = \frac{8}{2} = 4$$

So, $\lambda_1 = 6$ & $\lambda_2 = 4$

c) Trace Check.

$\text{trace}(M_1) = A + D = 4 + 6 = 10$

Sum of eigenvalues $= 6 + 4$
$= 10$

Equal!

Q2) $M_1 = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}$ w/ eigenvalues $\lambda_1 = 6, \lambda_2 = 4$

a) Eigenvalues.

For $\lambda = 6$

Solve $(M_1 - 6I)v = 0,$

$$M_1 - 6I = \begin{bmatrix} 4-6 & 0 \\ 9 & 6-6 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 \\ 9 & 0 \end{bmatrix}$$

Equations: $-2x = 0$ & $9x = 0 \Rightarrow x = 0$, y free.

Choose $y = 1$ An eigenvector: $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For $\lambda = 4$

Solve $(M_1 - 4I)v = 0$

$$M_1 - 4I = \begin{bmatrix} 0 & 0 \\ 9 & 2 \end{bmatrix}$$

Equations: $0\cdot x + 0\cdot y = 0$ (no info)
& $9x + 2y = 0 \Rightarrow y = -\frac{9}{2}x$

Choose $x = 2$

$$v_4 = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$$

Q2 continued.

b) Normalize.

For $v_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $\|V_6\| = \sqrt{0^2 + 1^2}$
$= 1$

Normalized $\hat{v}_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (already unit)

For $v_4 = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$: $\|v_4\| = \sqrt{2^2 + (-9)^2}$
$= \sqrt{85}$.

Normalized. $\hat{v}_4 = \frac{1}{\sqrt{85}} \begin{bmatrix} 2 \\ -9 \end{bmatrix}$

c) Verify $M_1 v = \lambda v$

For $\lambda = 6$, $V_6 = (0, 1)^T$:

$$M_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 0 + 0 \cdot 1 \\ 9 \cdot 0 + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $\lambda = 4$, $v_4 = (2, -9)^T$:

$$M_1 \begin{bmatrix} 2 \\ -9 \end{bmatrix} = \begin{bmatrix} 4 \cdot 2 + 0 \cdot (-9) \\ 9 \cdot 2 + 6 \cdot (-9) \end{bmatrix} = \begin{bmatrix} 8 \\ 18 - 54 \end{bmatrix} = \begin{bmatrix} 8 \\ -36 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ -9 \end{bmatrix}$$

Q3. Eigenvalues of 3x3 matrix

$$M_2 = \begin{bmatrix} A & 1 & 0 \\ 0 & B & 1 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

a) Characteristic polynomial.

For any triangular matrix T,

$$\det(\lambda I - T) = \prod_i (\lambda - t_{ii})$$

So

$$\chi M_2(\lambda) = (\lambda - A)(\lambda - B)(\lambda - C)$$

Substituting $A = 4$, $B = 0$, $C = 9$, $D = 6$

$$\chi M_2(\lambda) = (\lambda - 4)(\lambda - 0)(\lambda - 9)(\lambda - 6)$$
$$= \lambda^3 - 13\lambda^3 + 36\lambda$$

b) Eigenvalues.

Roots of χM_3 are the eigenvalues.

$\lambda = 4$, $\lambda = 0$, $\lambda = 9$

c) Key property of triangular matrices.

The eigenvalues of a triangular matrix are exactly its diagonal entries.

Equivalently,

$$\det(\lambda I - T) = \prod_i (\lambda - t_{ii})$$

so computing eigenvalues reduces to reading diagonal.

Q4. Eigenvectors of a 3×3 matrix.

$$M_3 = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 9 \end{bmatrix}$$

$$\det(\lambda I - M_3) = \begin{bmatrix} \lambda-4 & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda-9 \end{bmatrix}$$

Expanding along the first row.

$$\chi(\lambda) = (\lambda-4)\begin{vmatrix} \lambda & -1 \\ -1 & \lambda-9 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & \lambda-9 \end{vmatrix} + (-1)\begin{vmatrix} -1 & \lambda \\ -1 & -1 \end{vmatrix}$$

Compute each 2×2 determinant...

$$\det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda-9 \end{bmatrix} = \lambda(\lambda-9) - (-1)(-1) = \lambda^2 - 9\lambda - 1$$

$$\det\begin{bmatrix} -1 & -1 \\ -1 & \lambda-9 \end{bmatrix} = (-1)(\lambda-9) - (-1)(-1) = -\lambda + 8$$

$$\det\begin{bmatrix} -1 & \lambda \\ -1 & -1 \end{bmatrix} = (-1)(-1) - (\lambda)(-1) = 1 + \lambda$$

Substitute back.

$$\chi(\lambda) = (\lambda-4)(\lambda^2 - 9\lambda - 1) + (-\lambda + 8) - (1+\lambda)$$

Simplify

$$\chi(\lambda) = (\lambda-4)(\lambda^2 - 9\lambda - 1) - 2\lambda + 7$$

Expand.

$$(\lambda-4)(\lambda^2 - 9\lambda - 1) = \lambda^3 - 13\lambda^2 + 35\lambda + 4$$

Then.

$$\underline{\chi(\lambda) = \lambda^3 - 13\lambda^2 + 33\lambda + 11}$$

Solve.

$$\lambda^3 - 13\lambda^2 + 33\lambda + 11 = 0$$

No integer roots.

$\lambda_1 \approx 9.3412$

$\lambda_2 \approx 3.9564$

$\lambda_3 \approx -0.2976$.

b) Find the eigenvalues.

For each eigenvalue, solve $(M_3 - \lambda I)v = 0$

For $\lambda_1 = 9.3412$:

$$(M_3 - 9.3412I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 = \begin{cases} -5.3412x + y + z = 0 \\ x - 9.3412y + z = 0 \\ x + y - 0.3412z = 0. \end{cases}$$

Solving gives proportional components.

$x : y : z \approx 0.2053 : 0.1259 : 0.9706$

$$v_1 = \begin{bmatrix} 0.2053 \\ 0.1259 \\ 0.9706 \end{bmatrix}$$

For $\lambda_2 = 3.9564$

$$M_3 - 3.9564I \Rightarrow \begin{cases} 0.0436x + y + z = 0 \\ x - 3.9564y + z = 0 \\ x + y + 5.0436z = 0 \end{cases}$$

Solution proportional to $v_2 \approx (0.9564, 0.1846, -0.2262)^T$

For $\lambda_3 = -0.2976$

$$M_3 + 0.2976I = \begin{cases} 4.2976x + y + z = 0 \\ x + 0.2976y + z = 0 \\ x + y + 9.2976z = 0 \end{cases}$$

Solution proportional: $v_3 \approx (0.2076, -0.9747, 0.0825)^T$

Normalize to unit length...

$$\hat{v}_1 = \begin{bmatrix} 0.2053 \\ 0.1259 \\ 0.9706 \end{bmatrix} \quad \hat{v}_2 = \begin{bmatrix} 0.9564 \\ 0.1846 \\ -0.2262 \end{bmatrix} \quad \hat{v}_3 = \begin{bmatrix} 0.2076 \\ -0.9747 \\ 0.0825 \end{bmatrix}$$

c) Are the eigenvectors orthogonal.

Yes, because M_3 is real symmetric $(M_3^T = M_3)$

A real symmetric matrix always has real eigenvalues & mutually orthogonal eigenvectors for distinct eigenvalues.

Q5. Diagonalization & Verification.

$$M_4 = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}$$

a) Is M_4 diagonalizable?

Yes, the characteristic polynomial gave two distinct eigenvalues $\lambda_1 = 6$ & $\lambda_2 = 4$.

A 2×2 matrix with two distinct eigenvalues has two linearly independent eigenvectors, therefore, it is diagonalizable.

b) If yes, find P (eigenvectors)

Λ (eigenvalues)

such that $M4 = P \Lambda P^{-1}$

For $\lambda_1 = 6$, an eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 4$, an eigenvector $V_2 = \begin{pmatrix} 2 \\ -9 \end{pmatrix}$

Take P with these eigenvalues as columns & Λ the diagonal matrix of eigenvalues in the same order:

$$P = \begin{bmatrix} 0 & 2 \\ 1 & -9 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\det P = 0 \cdot (-9) - 2 \cdot 1 = -2$$

$$P^{-1} = \frac{1}{\det P}\begin{bmatrix} -9 & -2 \\ -1 & 0 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -9 & -2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

c) Verify $P\Lambda P^{-1} = M_4$

$$P\Lambda = \begin{bmatrix} 0 & 2 \\ 1 & -9 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0\cdot 6 & 2\cdot 4 \\ 1\cdot 6 & -9\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 8 \\ 6 & -36 \end{bmatrix}$$

multiply w/ P^{-1}

$$P\Lambda P^{-1} = \begin{bmatrix} 0 & 8 \\ 6 & -36 \end{bmatrix}\begin{bmatrix} \frac{9}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0\cdot\frac{9}{2} + 8\cdot\frac{1}{2} & 0\cdot 1 + 8\cdot 0 \\ 6\cdot\frac{9}{2} + (-36)\cdot\frac{1}{2} & 6\cdot 1 + (-36)\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4 & 0+0 \\ 27-18 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}$$

$$= M_4$$