Name: Thanawin Pattanaphol Assessment #1 - MAT209: 00010. ID: 013240096

Q1. Gaussian Elimination.

$$\begin{cases} 4x + 2y + 3z = 0 \\ 2x + 5y + z = 9 \\ 3x + 2y + 6z = 6 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 2 & 5 & 1 & -9 \\ 3 & 2 & 6 & -6 \end{array}\right]$$

Forward Elimination.

1. Eliminate x from Row2:

$$R_2 \leftarrow R_2 - \frac{2}{4}R_1 = R_2 - \frac{1}{2}R_1$$

$$R_2 = [2, 5, 1, 9] - \frac{1}{2}[4, 2, 3, 0] = [0, 4, -\frac{1}{2}, 6]$$

2. Eliminate x from Row3:

$$R_3 \leftarrow R_3 - \frac{3}{4}R_1$$

$$R_3 = [3, 2, 6, 6] - \frac{3}{4}[4, 2, 3, 0] = [0, \frac{1}{2}, \frac{15}{4}, 6]$$

Now the matrix is...

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 4 & -\frac{1}{2} & -9 \\ 0 & \frac{1}{2} & \frac{15}{4} & -6 \end{array}\right]$$

3. Eliminate y from Row3 using

$$R_3 \leftarrow R_3 - \left(\frac{1}{8}\right)R_2 \text{ (factor } = \frac{\frac{1}{2}}{4} = \frac{1}{8}\text{)}$$

$$R_3 = [0, \frac{1}{2}, \frac{15}{4}, 6] - \frac{1}{8}[0, 4, -\frac{1}{2}, 9]$$

$$= [0, 0, \frac{61}{16}, \frac{39}{8}]$$

Upper-triangular system...

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 4 & -\frac{1}{2} & -9 \\ 0 & 0 & \frac{61}{16} & -\frac{39}{8} \end{array}\right]$$

Back substitution.

Row3

$$\frac{61}{16}z = -\frac{39}{8} \Rightarrow z = \frac{\left(-\frac{39}{8}\right)}{\left(\frac{61}{16}\right)} = -\frac{78}{61}$$

Row2.

$$4y - \frac{1}{2}z = -9 \Rightarrow y = \frac{-9 + \frac{1}{2}z}{4} = \frac{-9 + \frac{1}{2}\left(-\frac{78}{61}\right)}{4}$$

$$= -\frac{147}{61}$$

Row1

$$4x + 2y + 3z = 0 \Rightarrow x = \frac{-2y - 3z}{4}$$

$$= \frac{-2 \cdot \frac{147}{61} - 3 \cdot \frac{78}{61}}{4}$$

$$= -\frac{132}{61}$$

$$\boxed{\begin{array}{l} \text{Therefore,} \\ x = -\frac{132}{61}, \\ y = \frac{147}{61}, \\ z = \frac{78}{61} \end{array}}$$

Q2) Augmented Matrix & Row Reduction.

$$\left[\begin{array}{ccc|c} 4 & 1 & 2 & 0 \\ 1 & 4 & 3 & 9 \\ 2 & 3 & 4 & 6 \end{array}\right]$$

a) Gaussian Elimination. (row operations)

$R_2' = 4 \cdot R_2 - 1R_1$

$= 4[1, 4, 3 | 9] - [4, 1, 2 | 0]$

$= [0, 15, 10 | 36]$

$R_3' = 4R_3 - 2R_1$

$= 4[2, 3, 4 | 6] - 2[4, 1, 2 | 0]$

$= [0, 10, 12 | 24]$

Eliminate y of R_3' w/ R_2'

Since,

$A^2 - 1 = 16 - 1$
$= 15$

$3A - 2 = 12 - 2$
$= 10$

So,

$R_3'' = 15 \cdot R_3' - 10 \cdot R_2'$

$= 15[0, 10, 12 | 24] - 10[0, 15, 10 | 36]$

$= [0, 0, 80 | 0]$

Upper-triangular augmented matrix...

$$\left[\begin{array}{ccc|c} 4 & 1 & 2 & 0 \\ 0 & 15 & 10 & 36 \\ 0 & 0 & 80 & 0 \end{array}\right]$$

b) Back substitution

Row 3.

$80z = 0 \Rightarrow z = 0$

Row 2.

$15y + 10z = 36 \Rightarrow 15y = 36$

$\Rightarrow y = \frac{36}{15} = \frac{12}{5}$

Row 1

$4x + y + 2z = 0 \Rightarrow 4x + \frac{12}{5} = 0$

$\Rightarrow x = -\frac{12}{5} \cdot \frac{1}{4}$

$= -\frac{3}{5}$

So...

$x = -\frac{3}{5}$ $z = 0$

$y = \frac{12}{5}$

Verification by substitution.

$4x + 1y + 2z = 4\left(-\frac{3}{5}\right) + \frac{12}{5} + 2 \cdot 0$
$= 0$
$= B$

$1x + 4y + 3z = -\frac{3}{5} + 4\left(\frac{12}{5}\right) + 0$
$= -\frac{3}{5} + \frac{48}{5}$
$= \frac{45}{5}$
$= 9$
$= C$

$2x + 3y + 4z = 2\left(-\frac{3}{5}\right) + 3\left(\frac{12}{5}\right) + 0$
$= -\frac{6}{5} + \frac{36}{5}$
$= \frac{30}{5}$
$= 6$
$= D$

Since

$(B, C, D) = (0, 9, 6)$ So solution is correct!

Q3) LU Decomposition

$$A = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} A+B \\ C+D \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \end{bmatrix}$$

a) Compute LU decomposition of A (no pivoting)

We seek $L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$ & $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$ with $A = LU$

$u_{11} = a_{11} = 4$ $\qquad l_{21} = \dfrac{a_{21}}{u_{11}} = \dfrac{9}{4}$

$u_{12} = a_{12} = 0$ $\qquad u_{22} = a_{22} - l_{21}u_{12} = 6 - \left(\frac{9}{4}\right) \cdot 0 = 6$

So

$$L = \begin{bmatrix} 1 & 0 \\ \frac{9}{4} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \text{ \& so } LU = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix} = A$$

b) Solve $Ax = b$ using LU

Solve $Ly = b$

$$\begin{bmatrix} 1 & 0 \\ \frac{9}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \end{bmatrix}$$

Forward substitution.

$y_1 = 4$

$y_2 = 15 - \frac{9}{4} \cdot y_1$

$= 15 - \frac{9}{4} \cdot 4$

$= 15 - 9$

$= 6$

Solve $Ux = y$

$$\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Back substitution.

$4x_1 = 4 \Rightarrow x_1 = 1$

$6x_2 = 6 \Rightarrow x_2 = 1$

So

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Verification.

$$Ax = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \end{bmatrix} = b$$

Correct!

Q4) Cholesky Decomposition.

$$A = \begin{bmatrix} A+4 & B \\ B & D+2 \end{bmatrix} = \begin{bmatrix} 4+4 & 0 \\ 0 & 6+2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

a) Verify if A is a positive definite.

Since... A is symmetric... & both principle minors are positive, so A is positive definite.

$m_1 = 8 > 0$

$\det(A) = 8 \cdot 8 - 0$

$= 64 > 0.$

b) If so, compute the Cholesky factorization $A = LL^T$

Since A is diagonal with equal entries, the Cholesky factor is diagonal with square roots on the diagonal.

$$L = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{8} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}$$

Check LL^T

$= \operatorname{diag}((\sqrt{8})^2, (\sqrt{8})^2) = \operatorname{diag}(8, 8)$

$= A.$

c) Solve $Ax = b$ using Cholesky

Solve $Ly = b$ then $L^T x = y$

1. $Ly = b$

$$\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix} \Rightarrow y_1 = 0, \quad y_2 = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}}$$

2. $L^T x = y$ (but $L^T = L$) here

$$\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{\sqrt{2}} \end{bmatrix} \Rightarrow x_1 = 0, \quad x_2 = \frac{\frac{3}{\sqrt{2}}}{2\sqrt{2}} = \frac{3}{4}$$

So

$$x = \begin{bmatrix} 0 \\ \frac{3}{4} \end{bmatrix}$$

Check:

$Ax = 8 \cdot [0, \frac{3}{4}]^T$

$= [0, 6]^T$

$= b$

Q5. Limit of matrix powers.

Let $M = \begin{bmatrix} 0.A & 0.B \\ 0.C & 0.D \end{bmatrix} = \begin{bmatrix} 0.4 & 0.0 \\ 0.9 & 0.6 \end{bmatrix}$

a) Compute the eigenvalues of M

Since M is lower triangular, its eigenvalues are the diagonal entries.

$\lambda_1 = 0.4, \quad \lambda_2 = 0.6.$

b) Determine whether $\lim (M^n)$ as $n \to \infty$ exists.

Yes, both eigenvalues have absolute value < 1.

For any square matrix, if $\rho(M) < 1$ then $M^n \to 0$ as $n \to \infty$.

So the limit exists.

c) The limit and why it converges.

Limit is zero matrix.

$$\lim_{n \to \infty} M^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $|\lambda_i| < 1$, powers of the eigenvalues go to zero; more generally $\|M^n\| \leq C \cdot \rho(M)^n$ for some constant C, so $M^n \to 0$.

Distinct eigenvalues here also imply M is diagonalizable, but diagonalizability is not required for the conclusion: spectral radius < 1 suffices.

Because M is lower triangular with a zero in the $(1,2)$ entry...

$$M^n = \begin{bmatrix} 0.4^n & 0 \\ 0.9 \sum_{k=0}^{n-1} 0.4^{n-1-k} 0.6^k & 0.6^n \end{bmatrix} = \begin{bmatrix} 0.4^n & 0 \\ 4.5(0.6^n - 0.4^n) & 0.6^n \end{bmatrix}$$

every entry tends to 0 as $n \to \infty$.

Bonus: QR Factorization.

$$A = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}$$

a) Compute via Gram-Schmidt

Let columns be

$$a_1 = \begin{bmatrix} 4 \\ 9 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Compute $u_1 = a_1$ & its norm

$$\|a_1\| = \sqrt{4^2 + 9^2} = \sqrt{16 + 81} = \sqrt{97}$$

Project a_2 into q_1:

$$\text{proj}_{q_1}(a_2) = (q_1^T a_2) q_1 \text{ with } q_1^T a_2 = \frac{1}{\sqrt{97}}(4 \cdot 0 + 9 \cdot 6) = \frac{54}{\sqrt{97}}$$

Thus

$$\text{proj}_{q_1}(a_2) = \frac{54}{\sqrt{97}} q_1 = \frac{54}{97}\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{216}{97} \\ \frac{486}{97} \end{bmatrix}$$

Form the second orthogonal vector.

$$u_2 = a_2 - \text{proj}_{q_1}(a_2) = \begin{bmatrix} 0 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{216}{97} \\ \frac{486}{97} \end{bmatrix} = \frac{1}{97}\begin{bmatrix} -216 \\ 96 \end{bmatrix} = \frac{24}{97}\begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

Its norm is.

$$\|u_2\| = \frac{24}{97}\sqrt{(-9)^2 + 4^2} = \frac{24}{97}\sqrt{97} = \frac{24}{\sqrt{97}}$$

So, second unit vector is

$$q_2 = \frac{u_2}{\|u_2\|} = \frac{\frac{24}{97}(-9, 4)^T}{\frac{24}{\sqrt{97}}} = \frac{1}{\sqrt{97}}\begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

The Q matrix.

Put q_1, q_2 as columns:

$$Q = (q_1, q_2) = \frac{1}{\sqrt{97}}\begin{bmatrix} 4 & -9 \\ 9 & 4 \end{bmatrix}$$

The R matrix.

Using $R = Q^T A$ or the Gram-Schmidt scalars.

$$r_{11} = \|a_1\| = \sqrt{97}; \; r_{12} = q_1^T a_2 = \frac{54}{\sqrt{97}}; \; r_{22} = \|u_2\| = \frac{24}{\sqrt{97}}$$

& $r_{21} = 0$ Thus,

$$R = \begin{bmatrix} \sqrt{97} & \frac{54}{\sqrt{97}} \\ 0 & \frac{24}{\sqrt{97}} \end{bmatrix} \xrightarrow{\text{Check.}} A = QR. \quad \frac{1}{\sqrt{97}}\begin{bmatrix} 4 & -9 \\ 9 & 4 \end{bmatrix}\begin{bmatrix} \sqrt{97} & \frac{54}{\sqrt{97}} \\ 0 & \frac{24}{\sqrt{97}} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix} = A$$

b) Verify $Q^T Q = I$

Compute.

$$Q^T Q = \frac{1}{97}\begin{bmatrix} 4 & 9 \\ -9 & 4 \end{bmatrix}\begin{bmatrix} 4 & -9 \\ 9 & 4 \end{bmatrix} = \frac{1}{97}\begin{bmatrix} 97 & 0 \\ 0 & 97 \end{bmatrix} = I_2$$

So Q is orthogonal.

c) Explain how QR can be used to solve $Ax = b$.

If $A = QR$ with orthogonal Q & upper-triangular invertible R, then.

$$Ax = b \Rightarrow QRx = b.$$

Multiply both sides by Q^T (since $Q^T Q = I$)

$$Rx = Q^T b$$

Because R is upper triangular, solve $Rx = Q^T$ by back substitution. (efficient & numerically stable)

Via..

- Compute $y = Q^T b$ then solve the triangular system $Rx = y$ for x via back-substitution