

Let A, B, C, D be last four digits of student ID. (4096)

So A = 4, B = 0, C = 9, D = 6

① Vector Operations.

Let $\vec{u} = (4, 0)$, $\vec{v} = (9, 6)$

a) $\vec{u} + \vec{v}$

$$\begin{aligned} \vec{u} + \vec{v} &= (4, 0) + (9, 6) \\ &= (4+9, 0+6) \\ &= (13, 6) \end{aligned}$$

b) $\vec{u} - \vec{v}$

$$\begin{aligned} \vec{u} - \vec{v} &= (4, 0) - (9, 6) \\ &= (4-9, 0-6) \\ &= (-5, -6) \end{aligned}$$

c) $3\vec{u}$

$$\begin{aligned} 3\vec{u} &= 3 \cdot (4, 0) \\ &= (3 \cdot 4, 3 \cdot 0) \\ &= (12, 0) \end{aligned}$$

② Dot Product & Norm

Let $\vec{u} = (4, 0)$, $\vec{v} = (9, 6)$

a) $\vec{u} \cdot \vec{v}$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (4, 0) \cdot (9, 6) \\ &= 4 \cdot 9 + 0 \cdot 6 \\ &= 36 + 0 \\ &= 36 \end{aligned}$$

b) $\|\vec{u}\|$ & $\|\vec{v}\|$

$$\begin{aligned} \|\vec{u}\| &= \|(4, 0)\| & \|\vec{v}\| &= \|(9, 6)\| \\ &= \sqrt{4^2 + 0^2} & &= \sqrt{9^2 + 6^2} \\ &= \sqrt{16 + 0} & &= \sqrt{81 + 36} \\ &= \sqrt{16} & &= \sqrt{117} \\ &= 4 & &\approx 10.817 \end{aligned}$$

c) Are they orthogonal?

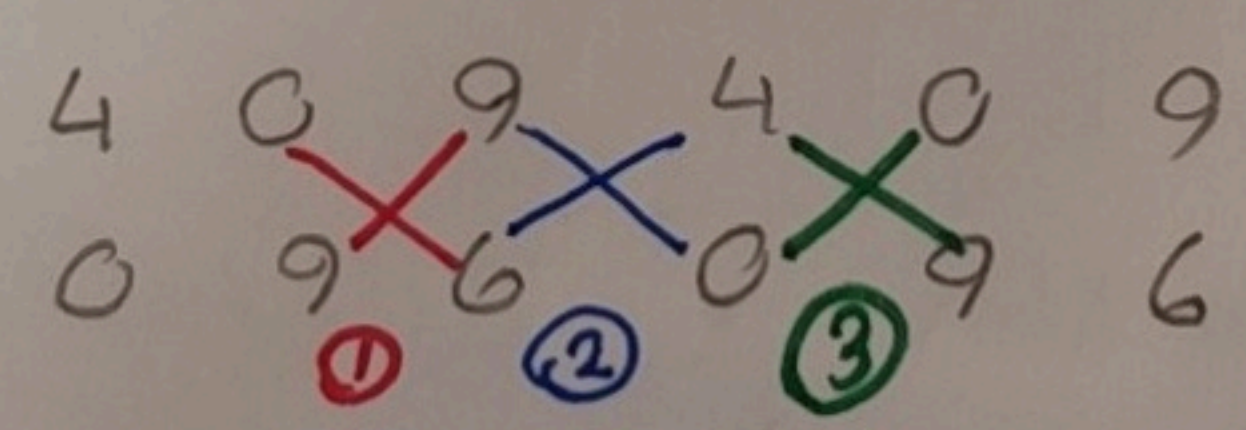
No. Because the dot product of \vec{u} & \vec{v} is not equal to 0

③ Cross Product.

Let $\vec{p} = (4, 0, 9)$, $\vec{q} = (0, 9, 6)$

a) $\vec{p} \times \vec{q}$

$$\begin{aligned} \vec{p} \times \vec{q} &= (4, 0, 9) \times (0, 9, 6) \\ &= (0-81, 0-24, 36-0) \\ &= (-81, -24, 36) \end{aligned}$$

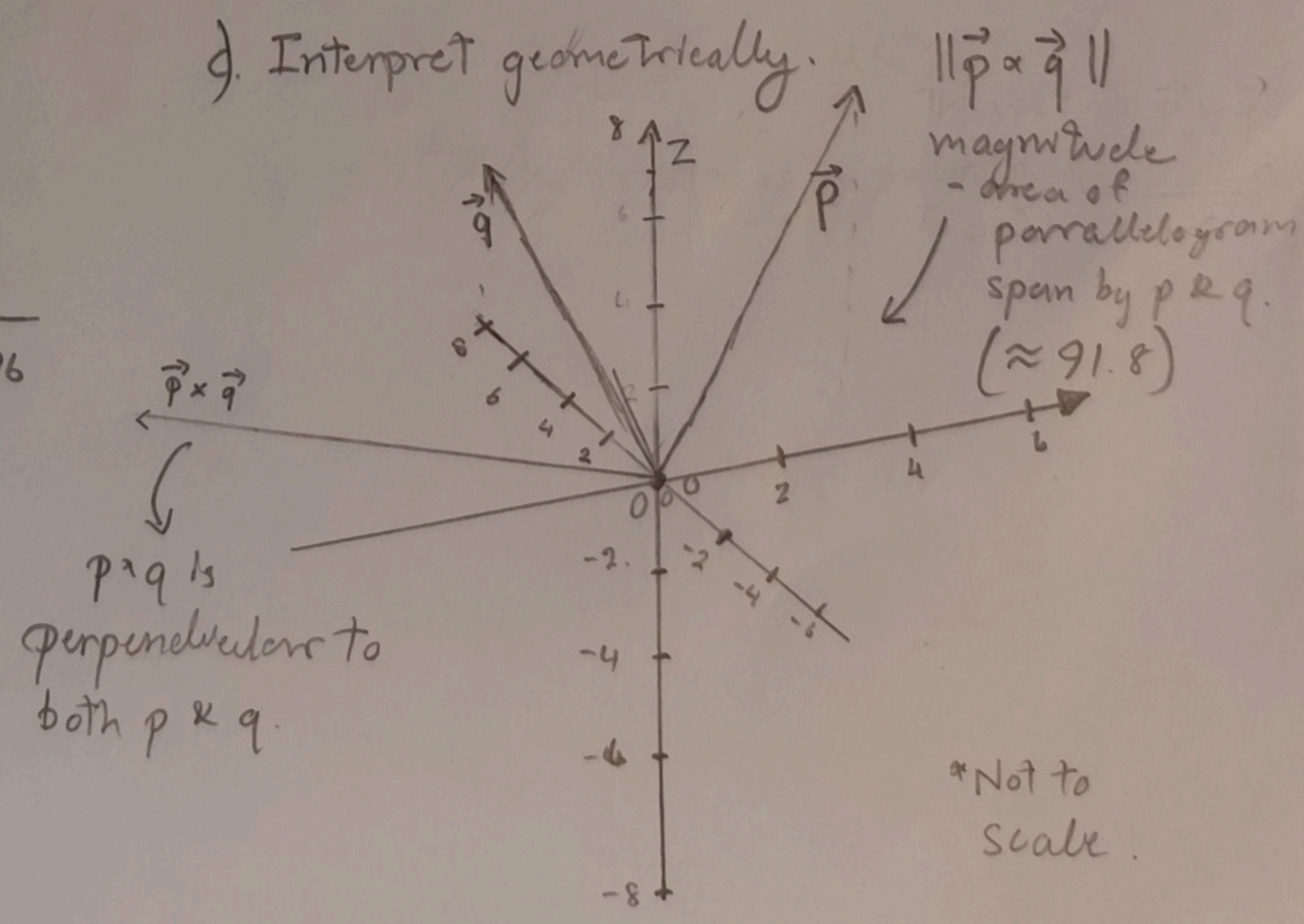


b) $\|\vec{p} \times \vec{q}\|$

Since $\vec{p} \times \vec{q} = (-81, -24, 36)$

$$\begin{aligned} \text{Then } \|\vec{p} \times \vec{q}\| &= \sqrt{(-81)^2 + (-24)^2 + (36)^2} \\ &= \sqrt{6561 + 576 + 1296} \\ &\approx 91.831 \text{ approx.} \end{aligned}$$

d) Interpret geometrically.



④ Matrices.

$$M = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}, N = \begin{bmatrix} 0 & 9 \\ 6 & 4 \end{bmatrix}$$

a) $M + N$

$$M + N = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 9 \\ 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0 & 0+9 \\ 9+6 & 6+4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 \\ 15 & 10 \end{bmatrix}$$

b) $M \cdot N$

$$M \cdot N = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} 0 & 9 \\ 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot 0 + 0 \cdot 6 & 4 \cdot 9 + 0 \cdot 4 \\ 9 \cdot 0 + 6 \cdot 6 & 9 \cdot 9 + 6 \cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 36 \\ 36 & 105 \end{bmatrix}$$

c) M^T

$$M^T = \begin{bmatrix} 4 & 9 \\ 0 & 6 \end{bmatrix}$$

⑤ Vector Space (Subspace)

Given $A = 4, B = 0$

a) Show $W = \{(x, y) \in \mathbb{R}^2 : 4x + 0 \cdot y = 0\} = \{(x, y) : 4x = 0\}$

So $4x = 0$

$\hookrightarrow x = 0, y = 0 \cdot A + B \cdot 0 = 0$

$\therefore W = \{(0, y) : y \in \mathbb{R}\}$

a) W is a subspace.

i) Zero vector.

The $(0, 0) \in W$

ii) Closed under addition.

$(0, y_1) + (0, y_2) = (0, y_1 + y_2) \in W$

iii) Closed under scalar multiplication.

$c(0, y) = (0, cy) \in W$

$\therefore W$ is a subspace of \mathbb{R}^2

b) $\dim(W)$

All vectors in W are multiples of $(0, 1)$

So $\dim(W) = 1$

c) Basis for W

A convenient basis would be $\{(0, 1)\}$ as it spans W & linearly independent.

⑥ Linear Combination & Span.

Let $u = (4, 0, 9)$

$v = (0, 9, 6)$

$w = (13, 6, 13)$

a) Try $w = \alpha u + \beta v$.

$$(4\alpha, 9\beta, 9\alpha + 6\beta) = (13, 6, 13)$$

From the first two equations $\alpha = \frac{13}{4}$, $\beta = \frac{6}{9} = \frac{2}{3}$. Check...

$$9\alpha + 6\beta = 9 \cdot \frac{13}{4} + 6 \cdot \frac{2}{3} =$$

$$= \frac{117}{4} + 4$$

$$= \frac{133}{4}$$

$$\neq 13.$$

\therefore No α, β satisfy all three equations, w cannot be written as $\alpha u + \beta v$

b) Is w in $\text{span}\{u, v\}$?

No, because third coordinate equation is not satisfied, no pair α, β can work.
 w cannot be expressed as $w \notin \text{span}\{u, v\}$.

c) Do $\{u, v\}$ span \mathbb{R}^3 ?

u & v are not scalar multiples of each other, $\text{span}\{u, v\}$ is a 2-dimensional plane.

SO, does not span \mathbb{R}^3

7) Let $a = (4, 0, 0)$, $b = (9, 6, 4)$, $c = (0, 9, 6)$

$$\text{Let } M = \begin{bmatrix} 4 & 9 & 0 \\ 0 & 6 & 9 \\ 0 & 4 & 6 \end{bmatrix}$$

a) Test if $\{a, b, c\}$ are independent?

$$\det M = 4(6 \cdot 6 - 9 \cdot 4) - 9(0 \cdot 6 - 9 \cdot 0) + 0$$

$$= 4(36 - 36)$$

$$= 0$$

Since determinant = 0, $\{a, b, c\}$ are independent.

b) Find a non-trivial relation.

$$(\alpha, \beta, \gamma) = \left(\frac{27}{8}, -\frac{3}{2}, 1\right)$$

$$27a - 12b + 8c = 0$$

Check:

$$27a = (108, 0, 0)$$

$$-12b = (-108, -72, -48)$$

$$8c = (0, 72, 48)$$

The sum is 0.

Therefore, $\{a, b, c\}$ is dependent, explicit dependence.

$$27a - 12b + 8c = 0$$

8) Given $M = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}$

a) $\det(M) = AD - BC$
 $= 4 \cdot 6 - 0 \cdot 9$
 $= 24$

b) Invertibility.
A 2×2 matrix is invertible
if $\det M \neq 0$
Since $24 \neq 0$, M is invertible.

c) Inverse
For a 2×2 matrix.

$$= M^{-1} = \frac{1}{AD - BC} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

$$= M^{-1} = \frac{1}{24} \begin{bmatrix} 6 & 0 \\ -9 & 4 \end{bmatrix}$$

$$= M^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & \frac{1}{6} \end{bmatrix}$$

9) Linear Transformation.

$$M = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix} \quad \det M = 4 \cdot 6 - 0 \cdot 9$$
$$= 24 > 0$$

a) $T(1, 0) = M(1, 0)$
 $= (A, C)$
 $= (4, 9)$

b) The unit square maps to the parallelogram
spanned by $(4, 9)$ and $(0, 6)$.

$$\text{Area} = |\det M| = 24.$$

Orientation is preserved since $\det M > 0$.

c)

$$T(A, D) = M \begin{bmatrix} A \\ D \end{bmatrix}$$
$$= \begin{bmatrix} A^2 + BD \\ AC + D^2 \end{bmatrix}$$

$$= \begin{bmatrix} 4^2 + 0 \cdot 6 \\ 9 \cdot 4 + 6^2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 72 \end{bmatrix}$$

⑩ Kernel & Image.

$$T(x) = Mx$$

$$\text{Let } M = \begin{bmatrix} 4 & 0 \\ 9 & 6 \end{bmatrix}$$

a) Show T is Linear.

$$\begin{aligned} T(u+v) &= M(u+v) \\ &= Mu + Mv \\ &= T(u) + T(v) \end{aligned}$$

$$\begin{aligned} T(\lambda u) &= M(\lambda u) \\ &= \lambda Mu \\ &= \lambda T(u). \end{aligned}$$

Thus, T is linear.

b) Find kernel of T .

$$\text{Solve } Mx = 0$$

$$\det M = 4 \cdot 6 - 0 \cdot 9 =$$

$$= 24$$

$$\neq 0$$

Since M is invertible

Therefore, $\ker T = \{0\}$.

c) $\dim(\text{Im } T)$

Since M is invertible

$$\text{rank } M = 2$$

Thus.

$$\dim(\text{Im } T) = 2.$$